c) $f(x) = 5x^{-2}$

$$f'(x) = 5(-2) \cdot x^{[-2-(1)]} = -10x^{-3} = -\frac{10}{x^3}$$

d) $y = -9x^{-4}$

$$\frac{dy}{dx} = -9(-4) \cdot x^{[-4-(1)]} = 36x^{-5} = \frac{36}{x^5}$$

e) $y = \frac{7}{x} = 7x^{-1}$

$$D_x(7x^{-1}) = 7(-1)x^{-2} = -7x^{-2} = -\frac{7}{x^2}$$

f) $f(x) = 18\sqrt{x} = 18x^{1/2}$

$$\frac{df}{dx} = 18\left(\frac{1}{2}\right) \cdot x^{1/2-1} = 9x^{-1/2} = \frac{9}{\sqrt{x}}$$

3.8. Use the rule for sums and differences to differentiate the following functions. Treat the dependent variable on the left as y and the independent variable on the right as x.

a) $R = 8t^2 + 5t - 6$

$$\frac{dR}{dt} = 16t + 5$$

b) $C = 4t^3 - 9t^2 + 28t - 68$

$$C' = 12t^2 - 18t + 28$$

c) $p = 6q^5 - 3q^3$

$$\frac{dp}{dq} = 30q^4 - 9q^2$$

d) $q = 7p^4 + 15p^{-3}$

$$D_p(7p^4 + 15p^{-3}) = 28p^3 - 45p^{-4}$$

THE PRODUCT RULE

3.9. Given $y = f(x) = 5x^4(3x - 7)$, (a) use the product rule to find the derivative. (b) Simplify the original function first and then find the derivative. (c) Compare the two derivatives.

a) Recalling the formula for the product rule from (3.3),

$$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

let $g(x) = 5x^4$ and $h(x) = 3x - 7$. Then $g'(x) = 20x^3$ and $h'(x) = 3$. Substitute these values in the product − rule formula.

$$y' = f'(x) = 5x^4(3) + (3x - 7)(20x^3)$$

Simplify algebraically.

$$y' = 15x^4 + 60x^4 - 140x^3 = 75x^4 - 140x^3$$

b) Simplify the original function by multiplication.

$$y = 5x^4(3x - 7) = 15x^5 - 35x^4$$

Take the derivative.

$$y' = 75x^4 - 140x^3$$

c) The derivatives found in parts (a) and (b) are identical. The derivative of a product can be found by either method, but as the functions grow more complicated, the product rule becomes more useful. Knowledge of another method helps to check answers.

3.10. Redo Problem 3.9, given $y = f(x) = (x^8 + 8)(x^6 + 11)$.

a) Let $g(x) = x^8 + 8$ and $h(x) = x^6 + 11$. Then $g'(x) = 8x^7$ and $h'(x) = 6x^5$. Substituting these values in *(3.3)*,

$$y' = f'(x) = (x^8 + 8)(6x^5) + (x^6 + 11)(8x^7)$$
$$= 6x^{13} + 48x^5 + 8x^{13} + 88x^7 = 14x^{13} + 88x^7 + 48x^5$$

b) Simplifying first through multiplication,

$$y = (x^8 + 8)(x^6 + 11) = x^{14} + 11x^8 + 8x^6 + 88$$

Then

$$y' = 14x^{13} + 88x^7 + 48x^5$$

c) The derivatives are identical.

3.11. Differentiate each of the following functions using the product rule. *Note*: The choice of problems is purposely kept simple in this and other sections of the book to enable students to see how various rules work. While it is proper and often easier to simplify a function algebraically before taking the derivative, applying the rules to the problems as given in the long run will help the student to master the rules more efficiently.

a) $y = (4x^2 - 3)(2x^5)$

$$\frac{dy}{dx}(4x^2 - 3)(10x^4) + 2x^5(8x) = 40x^6 - 30x^4 + 16x^6 = 56x^6 - 30x^4$$

b) $y = 7x^9(3x^2 - 12)$

$$\frac{dy}{dx} = 7x^9(6x) + (3x^2 - 12)(63x^8) = 42x^{10} + 189x^{10} - 756x^8 = 231x^{10} - 756x^8$$

c) $y = (2x^4 + 5)(3x^5 - 8)$

$$\frac{dy}{dx} = (2x^4 + 5)(15x^4) + (3x^5 - 8)(8x^3) = 30x^8 + 75x^4 + 24x^8 - 64x^3 = 54x^8 + 75x^4 - 64x^3$$

d) $z = (3 - 12t^3)(5 + 4t^6)$

$$\frac{dz}{dt} = (3 - 12t^3)(24t^5) + (5 + 4t^6)(-36t^2) = 72t^5 - 288t^8 - 180t^2 - 144t^8 = -432t^8 + 72t^5 - 180t^2$$

QUOTIENT RULE

3.12. Given

$$y = \frac{10x^8 - 6x^7}{2x}$$

(*a*) Find the derivative directly, using the quotient rule. (*b*) Simplify the function by division and then take its derivative. (*c*) Compare the two derivatives.

a) From *(3.4)*, the formula for the quotient rule is

$$f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

where $g(x)$ = the numerator = $10x^8 - 6x^7$ and $h(x)$ = the denominator = $2x$. Take the individual derivatives.

$$g'(x) = 80x^7 - 42x^6 \qquad h'(x) = 2$$

Substitute in the formula,

$$y' = \frac{2x(80x^7 - 42x^6) - (10x^8 - 6x^7)(2)}{(2x)^2}$$

$$= \frac{160x^8 - 84x^7 - 20x^8 + 12x^7}{4x^2} = \frac{140x^8 - 72x^7}{4x^2}$$

$$= 35x^6 - 18x^5$$

b) Simplifying the original function first by division,

$$y = \frac{10x^8 - 6x^7}{2x} = 5x^7 - 3x^6$$

$$y' = 35x^6 - 18x^5$$

c) The derivatives will always be the same if done correctly, but as functions grow in complexity, the quotient rule becomes more important. A second method is also a way to check answers.

3.13. Differentiate each of the following functions by means of the quotient rule. Continue to apply the rules to the functions as given. Later, when all the rules have been mastered, the functions can be simplified first and the easiest rule applied.

a) $y = \dfrac{3x^8 - 4x^7}{4x^3}$

Here $g(x) = 3x^8 - 4x^7$ and $h(x) = 4x^3$. Thus, $g'(x) = 24x^7 - 28x^6$ and $h'(x) = 12x^2$. Substituting in the quotient formula,

$$y' = \frac{4x^3(24x^7 - 28x^6) - (3x^8 - 4x^7)(12x^2)}{(4x^3)^2}$$

$$= \frac{96x^{10} - 112x^9 - 36x^{10} + 48x^9}{16x^6} = \frac{60x^{10} - 64x^9}{16x^6} = 3.75x^4 - 4x^3$$

b) $y = \dfrac{4x^5}{1 - 3x} \qquad (x \neq \frac{1}{3})$

(*Note*: The qualifying statement is added because if $x = \frac{1}{3}$, the denominator would equal zero and the function would be undefined.)

$$\frac{dy}{dx} = \frac{(1 - 3x)(20x^4) - 4x^5(-3)}{(1 - 3x)^2} = \frac{20x^4 - 60x^5 + 12x^5}{(1 - 3x)^2} = \frac{20x^4 - 48x^5}{(1 - 3x)^2}$$

c) $y = \dfrac{15x^2}{2x^2 + 7x - 3}$

$$\frac{dy}{dx} = \frac{(2x^2 + 7x - 3)(30x) - 15x^2(4x + 7)}{(2x^2 + 7x - 3)^2}$$

$$= \frac{60x^3 + 210x^2 - 90x - 60x^3 - 105x^2}{(2x^2 + 7x - 3)^2} = \frac{105x^2 - 90x}{(2x^2 + 7x - 3)^2}$$

d) $y = \dfrac{6x - 7}{8x - 5} \qquad (x \neq \frac{5}{8})$

$$\frac{dy}{dx} = \frac{(8x - 5)(6) - (6x - 7)(8)}{(8x - 5)^2} = \frac{48x - 30 - 48x + 56}{(8x - 5)^2} = \frac{26}{(8x - 5)^2}$$

e) $y = \dfrac{5x^2 - 9x + 8}{x^2 + 1}$

$$\frac{dy}{dx} = \frac{(x^2+1)(10x-9) - (5x^2-9x+8)(2x)}{(x^2+1)^2}$$
$$= \frac{10x^3 - 9x^2 + 10x - 9 - 10x^3 + 18x^2 - 16x}{(x^2+1)^2} = \frac{9x^2 - 6x - 9}{(x^2+1)^2}$$

THE GENERALIZED POWER FUNCTION RULE

3.14. Given $y = (5x+8)^2$, (*a*) use the generalized power function rule to find the derivative; (*b*) simplify the function first by squaring it and then take the derivative; (*c*) compare answers.

a) From the generalized power function rule in (*3.5*), if $f(x) = [g(x)]^n$,

$$f'(x) = n[g(x)]^{n-1} \cdot g'(x)$$

Here $g(x) = 5x + 8$, $g'(x) = 5$, and $n = 2$. Substitute these values in the generalized power function rule,

$$y' = 2(5x+8)^{2-1} \cdot 5 = 10(5x+8) = 50x + 80$$

b) Square the function first and then take the derivative,

$$y = (5x+8)(5x+8) = 25x^2 + 80x + 64$$
$$y' = 50x + 80$$

c) The derivatives are identical. But for higher, negative, and fractional values of n, the generalized power function rule is faster and more practical.

3.15. Find the derivative for each of the following functions with the help of the generalized power function rule.

a) $y = (6x^3 + 9)^4$

Here $g(x) = 6x^3 + 9$, $g'(x) = 18x^2$, and $n = 4$. Substitute in the generalized power function rule,

$$y' = 4(6x^3+9)^{4-1} \cdot 18x^2$$
$$= 4(6x^3+9)^3 \cdot 18x^2 = 72x^2(6x^3+9)^3$$

b) $y = (2x^2 - 5x + 7)^3$

$$y' = 3(2x^2 - 5x + 7)^2 \cdot (4x - 5)$$
$$= (12x - 15)(2x^2 - 5x + 7)^2$$

c) $y = \dfrac{1}{7x^3 + 13x + 3}$

First convert the function to an easier equivalent form,

$$y = (7x^3 + 13x + 3)^{-1}$$

then use the generalized power function rule,

$$y' = -1(7x^3 + 13x + 3)^{-2} \cdot (21x^2 + 13)$$
$$= -(21x^2 + 13)(7x^3 + 13x + 3)^{-2}$$
$$= \frac{-(21x^2 + 13)}{(7x^3 + 13x + 3)^2}$$

d) $y = \sqrt{34 - 6x^2}$

Convert the radical to a power function, then differentiate.

$$y = (34 - 6x^2)^{1/2}$$
$$y' = \tfrac{1}{2}(34 - 6x^2)^{-1/2} \cdot (-12x)$$
$$= -6x(34 - 6x^2)^{-1/2} = \frac{-6x}{\sqrt{34 - 6x^2}}$$

e) $y = \dfrac{1}{\sqrt{4x^3 + 94}}$

Convert to an equivalent form; then take the derivative.

$$y = (4x^3 + 94)^{-1/2}$$
$$y' = -\tfrac{1}{2}(4x^3 + 94)^{-3/2} \cdot (12x^2) = -6x^2(4x^3 + 94)^{-3/2}$$
$$= \frac{-6x^2}{(4x^3 + 94)^{3/2}} = \frac{-6x^2}{\sqrt{(4x^3 + 94)^3}}$$

CHAIN RULE

3.16. Use the chain rule to find the derivative dy/dx for each of the following functions of a function. Check each answer on your own with the generalized power function rule, noting that the generalized power function rule is simply a specialized use of the chain rule.

a) $y = (3x^4 + 5)^6$

Let $y = u^6$ and $u = 3x^4 + 5$. Then $dy/du = 6u^5$ and $du/dx = 12x^3$. From the chain rule in *(3.6)*,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Substituting,

$$\frac{dy}{dx} = 6u^5 \cdot 12x^3 = 72x^3u^5$$

But $u = 3x^4 + 5$. Substituting again,

$$\frac{dy}{dx} = 72x^3(3x^4 + 5)^5$$

b) $y = (7x + 9)^2$

Let $y = u^2$ and $u = 7x + 9$, then $dy/du = 2u$ and $du/dx = 7$. Substitute these values in the chain rule,

$$\frac{dy}{dx} = 2u \cdot 7 = 14u$$

Then substitute $7x + 9$ for u.

$$\frac{dy}{dx} = 14(7x + 9) = 98x + 126$$

c) $y = (4x^5 - 1)^7$

Let $y = u^7$ and $u = 4x^5 - 1$; then $dy/du = 7u^6$, $du/dx = 20x^4$, and

$$\frac{dy}{dx} = 7u^6 \cdot 20x^4 = 140x^4u^6$$

Substitute $u = 4x^5 - 1$.

$$\frac{dy}{dx} = 140x^4(4x^5 - 1)^6$$

3.17. Redo Problem 3.16, given:

a) $y = (x^2 + 3x - 1)^5$

Let $y = u^5$ and $u = x^2 + 3x - 1$, then $dy/du = 5u^4$ and $du/dx = 2x + 3$. Substitute in (*3.6*).

$$\frac{dy}{dx} = 5u^4(2x+3) = (10x + 15)u^4$$

But $u = x^2 + 3x - 1$. Therefore,

$$\frac{dy}{dx} = (10x + 15)(x^2 + 3x - 1)^4$$

b) $y = -3(x^2 - 8x + 7)^4$

Let $y = -3u^4$ and $u = x^2 - 8x + 7$. Then $dy/du = -12u^3$, $du/dx = 2x - 8$, and

$$\frac{dy}{dx} = -12u^3(2x - 8) = (-24x + 96)u^3$$
$$= (-24x + 96)(x^2 - 8x + 7)^3$$

COMBINATION OF RULES

3.18. Use whatever combination of rules is necessary to find the derivatives of the following functions. Do not simplify the original functions first. They are deliberately kept simple to facilitate the practice of the rules.

a) $y = \dfrac{3x(2x-1)}{5x-2}$

The function involves a quotient with a product in the numerator. Hence both the quotient rule and the product rule are required. Start with the quotient rule from (*3.4*).

$$y' = \frac{h(x)\cdot g'(x) - g(x)\cdot h'(x)}{[h(x)]^2}$$

where $g(x) = 3x(2x - 1)$, $h(x) = 5x - 2$, and $h'(x) = 5$. Then use the product rule from (*3.3*) for $g'(x)$.

$$g'(x) = 3x \cdot 2 + (2x - 1)\cdot 3 = 12x - 3$$

Substitute the appropriate values in the quotient rule.

$$y' = \frac{(5x-2)(12x-3) - [3x(2x-1)]\cdot 5}{(5x-2)^2}$$

Simplify algebraically.

$$y' = \frac{60x^2 - 15x - 24x + 6 - 30x^2 + 15x}{(5x-2)^2} = \frac{30x^2 - 24x + 6}{(5x-2)^2}$$

Note: To check this answer one could let

$$y = 3x \cdot \frac{2x-1}{5x-2} \qquad \text{or} \qquad y = \frac{3x}{5x-2}\cdot(2x-1)$$

and use the product rule involving a quotient.

b) $y = 3x(4x-5)^2$

The function involves a product in which one function is raised to a power. Both the product rule and the generalized power function rule are needed. Starting with the product rule,

$$y' = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

where $\qquad g(x) = 3x \qquad h(x) = (4x-5)^2 \qquad \text{and} \qquad g'(x) = 3$

Use the generalized power function rule for $h'(x)$.

$$h'(x) = 2(4x-5) \cdot 4 = 8(4x-5) = 32x - 40$$

Substitute the appropriate values in the product rule,

$$y' = 3x \cdot (32x - 40) + (4x-5)^2 \cdot 3$$

and simplify algebraically,

$$y' = 96x^2 - 120x + 3(16x^2 - 40x + 25) = 144x^2 - 240x + 75$$

c) $y = (3x-4) \cdot \dfrac{5x+1}{2x+7}$

Here we have a product involving a quotient. Both the product rule and the quotient rule are needed. Start with the product rule,

$$y' = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

where $\qquad g(x) = 3x - 4 \qquad h(x) = \dfrac{5x+1}{2x+7} \qquad \text{and} \qquad g'(x) = 3$

and use the quotient rule for $h'(x)$.

$$h'(x) = \frac{(2x+7)(5) - (5x+1)(2)}{(2x+7)^2} = \frac{33}{(2x+7)^2}$$

Substitute the appropriate values in the product rule,

$$y' = (3x-4) \cdot \frac{33}{(2x+7)^2} + \frac{5x+1}{2x+7} \cdot 3 = \frac{99x - 132}{(2x+7)^2} + \frac{15x+3}{2x+7}$$

$$= \frac{99x - 132 + (15x+3)(2x+7)}{(2x+7)^2} = \frac{30x^2 + 210x - 111}{(2x+7)^2}$$

One could check this answer by letting $y = (3x-4)(5x+1)/(2x+7)$ and using the quotient rule involving a product.

d) $y = \dfrac{(8x-5)^3}{(7x+4)}$

Start with the quotient rule, where

$$g(x) = (8x-5)^3 \qquad h(x) = 7x + 4 \qquad h'(x) = 7$$

and use the generalized power function rule for $g'(x)$,

$$g'(x) = 3(8x-5)^2 \cdot 8 = 24(8x-5)^2$$

Substitute these values in the quotient rule,

$$y' = \frac{(7x+4) \cdot 24(8x-5)^2 - (8x-5)^3 \cdot 7}{(7x+4)^2}$$

$$= \frac{(168x + 96)(8x-5)^2 - 7(8x-5)^3}{(7x+4)^2}$$

To check this answer, one could let $y = (8x-5)^3 \cdot (7x+4)^{-1}$ and use the product rule involving the generalized power function rule twice.

e) $y = \left(\dfrac{3x+4}{2x+5}\right)^2$

Start with the generalized power function rule,

$$y' = 2\left(\frac{3x+4}{2x+5}\right) \cdot \frac{d}{dx}\left(\frac{3x+4}{2x+5}\right) \tag{3.9}$$

Then use the quotient rule,

$$\frac{d}{dx}\left(\frac{3x+4}{2x+5}\right) = \frac{(2x+5)(3)-(3x+4)(2)}{(2x+5)^2} = \frac{7}{(2x+5)^2}$$

and substitute this value in (*3.9*),

$$y' = 2\left(\frac{3x+4}{2x+5}\right) \cdot \frac{7}{(2x+5)^2} = \frac{14(3x+4)}{(2x+5)^3} = \frac{42x+56}{(2x+5)^3}$$

To check this answer, let $y = (3x+4)^2 \cdot (2x+5)^{-2}$, and use the product rule involving the generalized power function rule twice.

3.19. Differentiate each of the following, using whatever rules are necesary:

a) $y = (5x-1)(3x+4)^3$

Using the product rule together with the generalized power function rule,

$$\frac{dy}{dx} = (5x-1)[3(3x+4)^2(3)] + (3x+4)^3(5)$$

Simplifying algebraically,

$$\frac{dy}{dx} = (5x-1)(9)(3x+4)^2 + 5(3x+4)^3 = (45x-9)(3x+4)^2 + 5(3x+4)^3$$

b) $y = \dfrac{(9x^2-2)(7x+3)}{5x}$

Using the quotient rule along with the product rule,

$$y' = \frac{5x[(9x^2-2)(7)+(7x+3)(18x)] - (9x^2-2)(7x+3)(5)}{(5x)^2}$$

Simplifying algebraically,

$$y' = \frac{5x(63x^2-14+126x^2+54x) - 5(63x^3+27x^2-14x-6)}{25x^2} = \frac{630x^3+135x^2+30}{25x^2}$$

c) $y = \dfrac{15x+23}{(3x+1)^2}$

Using the quotient rule plus the generalized power function rule,

$$y' = \frac{(3x+1)^2(15) - (15x+23)[2(3x+1)(3)]}{(3x+1)^4}$$

Simplifying algebraically,

$$y' = \frac{15(3x+1)^2 - (15x+23)(18x+6)}{(3x+1)^4} = \frac{-135x^2-414x-123}{(3x+1)^4}$$

d) $y = (6x+1)\dfrac{4x}{9x-1}$

Using the product rule and the quotient rule,

$$D_x = (6x+1)\frac{(9x-1)(4) - 4x(9)}{(9x-1)^2} + \frac{4x}{9x-1}(6)$$

Simplifying algebraically,

$$D_x = \frac{(6x+1)(36x-4-36x)}{(9x-1)^2} + \frac{24x}{9x-1} = \frac{216x^2 - 48x - 4}{(9x-1)^2}$$

e) $y = \left(\dfrac{3x-1}{2x+5}\right)^3$

Using the generalized power function rule and the quotient rule,

$$y' = 3\left(\frac{3x-1}{2x+5}\right)^2 \frac{(2x+5)(3) - (3x-1)(2)}{(2x+5)^2}$$

Simplifying algebraically,

$$y' = \frac{3(3x-1)^2}{(2x+5)^2}\,\frac{17}{(2x+5)^2} = \frac{51(3x-1)^2}{(2x+5)^4}$$

HIGHER-ORDER DERIVATIVES

3.20. For each of the following functions, (1) find the second-order derivative and (2) evaluate it at $x = 2$. Practice the use of the different second-order notations.

a) $y = 7x^3 + 5x^2 + 12$

1) $\dfrac{dy}{dx} = 21x^2 + 10x$

$\dfrac{d^2y}{dx^2} = 42x + 10$

2) At $x = 2$, $\dfrac{d^2y}{dx^2} = 42(2) + 10 = 94$

b) $f(x) = x^6 + 3x^4 + x$

1) $f'(x) = 6x^5 + 12x^3 + 1$

$f''(x) = 30x^4 + 36x^2$

2) At $x = 2$, $f''(x) = 30(2)^4 + 36(2)^2 = 624$

c) $y = (2x+3)(8x^2-6)$

1) $Dy = (2x+3)(16x) + (8x^2-6)(2)$

$= 32x^2 + 48x + 16x^2 - 12$

$= 48x^2 + 48x - 12$

$D^2y = 96x + 48$

2) At $x = 2$, $D^2y = 96(2) + 48 = 240$

d) $f(x) = (x^4-3)(x^3-2)$

1) $f' = (x^4-3)(3x^2) + (x^3-2)(4x^3)$

$= 3x^6 - 9x^2 + 4x^6 - 8x^3$

$= 7x^6 - 8x^3 - 9x^2$

$f'' = 42x^5 - 24x^2 - 18x$

2) At $x = 2$, $f'' = 42(2)^5 - 24(2)^2 - 18(2) = 1212$